9.8 Orthogonality theorems

We have just seen that if we know a systems irreps we can use them to block diagonalize a Hamiltonian. But we still don't have all the theoretical tools we need to identify irreps in the first place. We will set some of these out in this subsection.

9.8.1 Grand Orthogonality Theorem

We are now in a position to state the grand orthogonality theorem. Similarly to how the orthogonality of eigenstates of a Hermitian operator allows you to find a single eigenstate and then identify other eigenstates by construction, we will see that this theorem allows us to take one irrep and identify others by this orthogonality constraint.

We can think of irreducible representations as giving "vectors of matrices" $([R(g)]_{ij})_{g\in G}$ in a vector space of dimension $|G|$. The Grand Orthogonality Theorem provides orthogonality relations between these vectors. Let me start by stating the theorem in its full glory:

Theorem 9.8.1 (Grand Orthogonality Theorem)**.** *Let R_a and R_b be two non-equivalent unitary irreducible representations of a finite*[19] *group G of order N. Let n_a and n_b be the dimensions of the vector space for R_a and R_b. Then the grand orthogonality theorem states that*

$$\sum_{g\in G} \frac{n_a}{N} \left[R_a(g)^\dagger\right]_{jk} [R_b(g)]_{lm} = \delta_{ab}\delta_{jm}\delta_{lk} \tag{9.53}$$

The grand orthogonality theorem is a consequence of Schur's lemma, for a derivation see Appendix 9.13.

Now let me try and unpick it a little for you. Let's first consider the case of two non-equivalent irreps (i.e, $a \neq b$). Then the grand orthogonality theorem implies that the vectors of matrices corresponding to any two non-equivalent irreps are orthogonal[20]. In particular, we have

$$\sum_{g\in G} \left[R_a(g)^\dagger\right]_{jk} [R_b(g)]_{lm} = 0, \forall \ a \neq b, \forall i, j, k, l\,. \tag{9.54}$$

Next let's consider the case where $a = b$ so that we're just looking at the properties of a single irrep. In this case we firstly have an orthogonality relation between the elements of the irreps

$$\sum_{g\in G} \left[R_a(g)^\dagger\right]_{jk} [R_a(g)]_{lm} = 0 \text{ if } j \neq m \text{ and/or } l \neq k\,. \tag{9.55}$$

Finally, the grand orthogonality theorem provides a normalisation condition for these vectors in the case where $j = m$ and $l = k$. Concretely, we have

$$\sum_{g\in G} [R_a(g)^*]_{kj} [R_a(g)]_{kj} = \frac{N}{n_a}\,. \tag{9.56}$$

where N is the order of group G and n_a is the dimension of the vector space of representation R_a.

[19] The theorem can also be generalized to compact Lie groups.

[20] Note, that in fact the condition the Grand Orthogonality Theorem imposes is stronger than simply the orthogonality of these vectors. That would be the claim that $\sum_g R_a(g)^\dagger R_b(g) = 0$ which is equivalent to $\sum_g \sum_j [R_a(g)^\dagger]_{ij} [R_b(g)]_{jk} = 0$ for all i and k. This is implied by Eq.(9.54) but Eq.(9.54) is stronger.

Examples. As ever, let us try and make this a little less abstract by considering some examples. Let us start with the Z_2 group. It is Abelian so its irreps are one-dimensional. Specifically, we have:

$$R_1(e) = 1, R_1(a) = 1 \tag{9.57}$$
$$R_2(e) = 1, R_2(a) = -1\,. \tag{9.58}$$

As these are one-dimensional irreps we can drop the subscripts j, k, l, m in Eq. (9.54) and have:

$$\sum_g R_1(g)^\dagger R_2(g) = R_1(e)^\dagger R_2(e) + R_1(a)^\dagger R_2(a) = 1\times 1 + 1\times(-1) = 0 \tag{9.59}$$

in agreement with Eq. (9.54). Similarly,

$$\begin{aligned}\sum_g R_1(g)^\dagger R_1(g) &= 1\times 1 + 1\times 1 = 2\\ \sum_g R_2(g)^\dagger R_2(g) &= 1\times 1 + -1\times(-1) = 2\,.\end{aligned} \tag{9.60}$$

As the order of the group is 2 ($N = 2$) and the dimension of the irreps are 1 ($n_A = 1$) this agrees with Eq. (9.56).

As a less trivial example, let's consider C3v. Remember, this consisting of two rotations (clockwise and anti-clockwise) and three reflections (on each axis). A possible irreducible representation [21] are the following six real matrices:

$$\begin{aligned}&e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &c_+ = \begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix},\ c_- = \begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\\ &\sigma = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\ \sigma' = \begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix},\ \sigma'' = \begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\end{aligned} \tag{9.61}$$

Let us consider an example of the normalisation condition first:

$$\sum_{g\in G} R^\dagger(g)_{11} R(g)_{11} = 1^2 + 1^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = 3 = \frac{6}{2}.$$

which satisfies Eq. (9.56) as the order of the group is 6 ($N = 6$) and the dimension of the irrep is 2 ($n_A = 2$). Now let's demonstrate the orthogonality of the $(1, 1)$ and $(2, 2)$ elements:

$$\sum_{g\in G} R(g)^\dagger_{11} R(g)_{22} = 1^2 + (1)(-1) + \left(-\frac{1}{2}\right)\frac{1}{2} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\frac{1}{2} = 0.$$

It is straightforward to verify the orthogonality of the other elements.

A direct consequence of the grand orthogonality theorem is that

Proposition 9.8.2. *A finite group can only have a finite number of inequivalent irreducible representations. Specifically, the maximum number of possible irreps is given by the order of the group.*

[21] We will discuss how to check that this is indeed an irrep and discuss other irreps of C3v in Section 9.9.1

This is clear from the orthogonality theorem. Thinking of irreducible representations as giving "vectors of matrices" $([R(g)]_{ij})_{g\in G}$ in a vector space of dimension $|G|$, the theorem tells us that those vectors must be orthogonal. But there are at most $|G|$ orthogonal vectors in a vector space of dimension $|G|$, and so the number of irreducible representations must be finite. In fact, we will calculate the number of irreducible representations for any finite group explicitly when we introduce characters.

9.8.2 Group averaging (twirling)

You may have noticed that the grand orthogonality theorem looks a lot like an average of an object under the adjoint action of the group. To see this consider the quantity:

$$\langle X\rangle_G := \frac{1}{N}\sum_g R(g)XR(g)^\dagger\,. \tag{9.62}$$

For example, if $R(g) = U_g$ is a unitary representation then this is just the average output of X after being evolved by each unitary U_g in the group,

$$\langle X\rangle_G := \frac{1}{N}\sum_g U_g X U_g^\dagger\,. \tag{9.63}$$

If this representation is irreducible then we can apply the grand orthogonality theorem to get the following **Irrep Group Averaging Corollary**:

$$\begin{aligned}\langle X\rangle_G &= \frac{1}{N}\sum_{jklm}\sum_g [R(g)]_{lm}X_{mj}[R(g)^\dagger]_{jk}|l\rangle\langle k|\\ &= \frac{1}{d}\sum_{jklm}\delta_{lk}\delta_{jm}X_{mj}|l\rangle\langle k|\\ &= \frac{1}{d}\sum_{jk}X_{jj}|k\rangle\langle k|\\ &= \frac{1}{d}\operatorname{Tr}[X]\,I\end{aligned} \tag{9.64}$$

where $n_a = d$ is the dimension of the vector space of the representation.

Let's consider the group average of the single qubit Pauli group $G = \{\pm(i)\sigma_x, \pm(i)\sigma_y, \pm(i)\sigma_z, \pm(i)I\}$ over an arbitrary single qubit initial state ρ. This is an irreducible representation onto a $d = 2$ vector space and so from Eq. (9.64) we should have

$$\langle\rho\rangle_G = \frac{I}{2}\,. \tag{9.65}$$

That is, averaging the effect of applying each of the Paulis on a given state gives a maximally mixed state.

If it helps to make this less abstract and mysterious we can also compute $\langle\rho\rangle_G$ explicitly. To do so we first note that in each term of the form $U_g\rho U_g^\dagger$ the +1, -1, $+i$, $-i$ signs cancel out, i.e. $(i\sigma_z)\rho(-i\sigma_z) = \sigma_z\rho\sigma_z$, and so we can write

$$\langle\rho\rangle_G = \frac{1}{4}(\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z + I\rho I)\,. \tag{9.66}$$

If we write the state in terms of its Bloch vector, $\rho = \frac{1}{2}(I + \boldsymbol{r}.\boldsymbol{\sigma})$ and remember the properties of Pauli matrices (e.g. $\sigma_i\sigma_j\sigma_i = -\sigma_j$ for $i \neq j$ but $\sigma_j^3 = \sigma_j$) then we have

$$\langle\rho\rangle_G = \frac{1}{2}\left(I + \frac{1}{4}\left(\begin{pmatrix} r_x \\ -r_y \\ -r_z \end{pmatrix} + \begin{pmatrix} -r_x \\ r_y \\ -r_z \end{pmatrix} + \begin{pmatrix} -r_x \\ -r_y \\ r_z \end{pmatrix} + \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}\right).\boldsymbol{\sigma}\right) = \frac{1}{2}I\,, \tag{9.67}$$

in agreement with Eq. (9.65)

All this discussion of orthogonality theorems so far (i.e., both the grand orthogonality theorem and the group averaging corollary) has been framed for finite groups; however, it also carries over to compact (i.e. closed and bounded) Lie groups. And all the continuous groups we normally care about $U(n)$, $SU(n)$, $O(n)$, $SO(n)$ etc are compact. In this case the finite average sum $\frac{1}{N}\sum_g$ becomes a continuous integral over a uniform measure $\int d\mu(g)$. This uniform measure is called the Haar measure and the average is called Haar averaging - it's exact form and properties are beyond this course but I highly recommend this blog or this review. In any case, for continuous groups the average over irreducible representations is given by:

$$\langle X\rangle_G := \int_G d\mu(g)U_x(g)XU_x(g)^\dagger = \frac{1}{d}\,\mathrm{Tr}[X]\,I\,. \tag{9.68}$$

The operator $\int_G d\mu(g)U_x(g)...U_x(g)^\dagger$ is sometimes called the *twirling* operation[22].

For example, if you apply random unitaries to a single qubit state and then average the states you get out you will end up with the maximally mixed state. Note you effectively saw this in the decoherence problem sheet - but then I was nice and made the calculation simpler and had you just average over a mix of rotations around the σ_z and σ_x axes rather than arbitrary unitaries.

If you think back to the decoherence problem sheet you'll remember that if you only averaged over $R_z(\theta) = e^{-i\theta\sigma_z}$ rotations then you ended up not at the maximally mixed state but on projecting the state onto the Z axis. How can we understand this?

The first thing to note is that we cannot directly apply Eq. (9.68) because that only holds for irreps and $R_z(\theta) = e^{-i\theta\sigma_z}$ is not an irrep. To see this note that here we are considering U(1) which is an Abelian group and so all its irreps are 1D. So we need a generalization of Eq. (9.68) for reducible representations.

Any reducible unitary representation can be written in the form

$$U(g) = \bigoplus_x U_x(g)\,. \tag{9.69}$$

Let us consider a basis $\mathcal{B}_x = \{|x,i\rangle\}_{i=1}^{d_x}$ for each subspace x of dimension d_x. Therefore, $\bigcup_x \mathcal{B}_x$ is a basis for the full space (i.e. on which $U(g)$ acts) and we have

$$U(g) = \bigoplus_x U_x(g) = \sum_x \sum_{i,j=1}^{d_x} (U_x(g))_{i,j}\,|x,i\rangle\langle x,j|\,, \tag{9.70}$$

where $(U_x(g))_{i,j}$ is the component (i,j) of $U_x(g)$ with respect to the elements of $\mathcal{B}_x$ i.e. $(U_x(g))_{i,j} = \langle x,i|\,U(g)\,|x,j\rangle$. Let us repeat the calculation in Eq. (9.64) but this consider a

[22] In a quantum information context it is such standard terminology that I thought everyone called it this. However, apparently not... which lead to a few awkward conversations before I realised this.

reducible representation written as in Eq. (9.70). Again we'll do this calculation for a finite group but it generalises to continuous groups. Thus if we use the grand orthogonality theorem to repeat the calculation in Eq. (9.64) we find:

$$\begin{aligned}
\langle X\rangle_G &= \frac{1}{N}\sum_g U(g)XU(g)^\dagger \\
&= \frac{1}{N}\sum_g\sum_{xx'}\sum_{i,j=1}^{d_x}\sum_{k,l=1}^{d_{x'}} (U_x(g))_{i,j}\langle x,j|X|x',k\rangle(U_{x'}(g)^\dagger)_{k,l}|x,i\rangle\langle x',l| \\
&= \sum_{xx'}\sum_{i,j=1}^{d_x}\sum_{k,l=1}^{d_{x'}}\langle x,j|X|x',k\rangle|x,i\rangle\langle x',l|\underbrace{\frac{1}{N}\sum_g(U_{x'}(g)^\dagger)_{k,l}(U_x(g))_{i,j}}_{=\frac{\delta_{xx'}\delta_{il}\delta_{jk}}{d_{x'}}} \\
&= \sum_x\frac{1}{d_x}\sum_{i,j=1}^{d_x}\langle x,j|X|x,j\rangle|x,i\rangle\langle x,i| \\
&= \sum_x\frac{1}{d_x}\sum_{j=1}^{d_x}\langle x,j|X|x,j\rangle\Pi_x \\
&= \sum_x\frac{\mathrm{Tr}[\Pi_x X]}{d_x}\Pi_x \\
&= \bigoplus_x\frac{\mathrm{Tr}[\Pi_x X]}{d_x}I_x\ ,
\end{aligned} \tag{9.71}$$

where Π_x is the projector onto subspace x and I_x is the identity in this subspace ($\dim(I_x) = d_x$ and $\dim(\Pi_x) = \dim(X)$). The grand orthogonality theorem is used in the fourth equality, we perform the sum over i in the fifth inequality by introducing Π_x, then we recognise a trace over the projection on X onto subspace x (i.e. $\mathrm{Tr}[\Pi_x X\Pi_x] = \mathrm{Tr}[\Pi_x X]$ by cyclicity of the trace and as $\Pi_x^2 = \Pi_x$ for projector). (As a sanity check note that if we are actually looking at an irrep then we have $\Pi_x = I$ and $\mathrm{Tr}_x = \mathrm{Tr}$ and so Eq. (9.71) reduces to Eq. (9.64)). Again, while I have worked through this calculation for a finite group it also carries over to averaging over all the standard continuous groups we are interested in.

Ok so what happens when we average a state ρ by $R_z(\theta) = e^{-i\theta\sigma_z}$? Well the relevant group here is $U(1)$ and so the irreps in this case are both 1D ($\{1\}$ and $\{e^{-i\theta\theta}\}$) and we have:

$$U_g == \begin{pmatrix}1 & 0\\ 0 & e^{-i\theta}\end{pmatrix} = |0\rangle\langle 0| + e^{-i\theta}|1\rangle\langle 1| \tag{9.72}$$

such that $\Pi_0 = |0\rangle\langle 0|$, $\Pi_1 = |1\rangle\langle 1|$ and $I_1, I_2 = 1$

$$\langle\rho\rangle_G = \bigoplus_{x=0,1}\mathrm{Tr}[\rho\Pi_x] = \sum_{x=0,1}\mathrm{Tr}[\rho\Pi_x]\Pi_x = \langle 0|\rho|0\rangle|0\rangle\langle 0| + \langle 1|\rho|1\rangle|1\rangle\langle 1|\,. \tag{9.73}$$

Thus as we expected (inline with Problem Sheet 5) this averaging kills off all coherence and projects onto the Z axis. For a visualisation of the effect of twirling on the Bloch sphere see Fig. 9.6.

Exercise: What happens if you twirl a qubit state over the group $SU(2)\otimes SU(2)$?

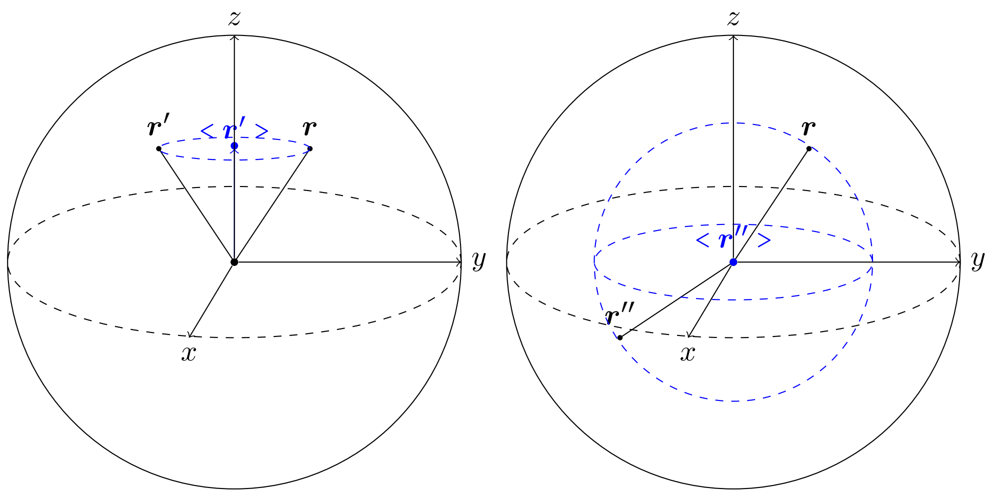

Figure 9.6: Left: We want the average of state $\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})$ by $R_z(\theta)$ where $\boldsymbol{r} = (r_x, r_y, r_z)$. If we rotate ρ around the z-axis it goes to $\rho' = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}'\cdot\boldsymbol{\sigma})$ where $\boldsymbol{r}' = (r'_x, r'_y, r_z)$. So if we calculate the average it would be a density matrix with a vector in the Block sphere equal to $(0, 0, r_z)$ which is along the z-axis. Right: And when we have all Pauli matrices, it will be an arbitrary rotation. So the state $\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})$ rotates and goes to $\rho'' = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}''\cdot\boldsymbol{\sigma})$ where $\boldsymbol{r}'' = (r''_x, r''_y, r''_z)$ is another arbitrary vector. Then the average is a density matrix with vector zero in the Block sphere.

9.8.3 Petit Orthogonality Theorem.

We just saw that the grand orthogonality theorem is effectively an orthogonality relation between "vectors of matrices" $([R(g)]_{ij})_{g\in G}$. We will now consider the petite orthogonality theorem, its simpler corollary, which is an orthogonality relation between vectors composed of their traces $(\chi_R(g))_{g\in G}$ where we have defined

$$\chi_R(g) \coloneqq \mathrm{Tr}[R(g)]\,. \tag{9.74}$$

We further note that $\mathrm{Tr}(R(x)^\dagger) = \chi^*_R(x)$.

Theorem 9.8.3 (Classes & Traces)**.** *In a representation R, all the elements which are in the same conjugacy class have the same trace.*

Demo. If there exists u such that $x = u^{-1}yu$ then

$$\begin{aligned}\mathrm{Tr}(R(x)) &= \mathrm{Tr}(R(u^{-1}yu)) = \mathrm{Tr}(R(u^{-1})R(y)R(u)) = \mathrm{Tr}(R(u)R(u^{-1})R(y)) = \mathrm{Tr}(R(e)R(y)) \\ &= \mathrm{Tr}(R(y))\end{aligned} \tag{9.75}$$

□

From the Grand Orthogonality Theorem, we find

$$\sum_{jk}\sum_{g\in G}\frac{n_a}{N}\left[R_a(g)^\dagger\right]_{jj}\left[R_b(g)\right]_{kk} = \sum_{g\in G}\frac{n_a}{N}\chi^*_a(g)\chi_b(g) = \delta_{ab}\sum_{jk}\delta_{jk}\delta_{jk} = n_a\delta_{ab} \tag{9.76}$$

where in the final line we use the fact that $\sum_{j,k=1}^{n_A}\delta_{j,k}\delta_{jk} = \sum_{j,k=1}^{n_A}\delta_{jk} = n_a$. Thus we see that the vectors of traces of two irreps are orthogonal. Or more formally:

Figure 9.7: **Motivational cat.** Here's also a link to one of my favourite cat videos. It's an old one, and a slow burner (from an era pre-tiktok when videos could be more than 60 seconds).

Theorem 9.8.4 (Petit Orthogonality Theorem)**.** *Let R_a and R_b denote two non-equivalent unitary irreducible representations of a finite group of order N, we have*

$$\sum_{g\in G} \chi_a^*(g)\chi_b(g) = N\delta_{a,b} \tag{9.77}$$

As elements in a conjugacy class have the same trace, one can equivalently write the petit orthogonality theorem by summing over the number of the conjugacy classes, i.e. we have

$$\sum_{\mu=1}^{N_c} n_\mu \chi_a^*(C_\mu)\chi_b(C_\mu) = N\delta_{a,b} \tag{9.78}$$

where n_μ denotes the number of elements in class μ and N_c is the total number of conjugacy classes.

For example, in the case of C3v we have three equivalent classes: $\{e\}$,$\{c_+, c_-\}$, and the three mirrors $\{\sigma, \sigma', \sigma''\}$. We see in Eq. (9.61) that $\chi(e) = 2$, $\chi(c_+) = \chi(c_-) = -1$ and $\chi(\sigma) = \chi(\sigma') = \chi(\sigma'') = 0$. Thus, in line with Eq. (9.78), we have $1 \times 2^2 + 2 \times (-1)^2 + 3 \times 0^2 = 6$.

We stress that we can interpret this theorem as an orthogonality relation of N_r (the number of representations) vectors in a space of dimension N_c (the number of equivalent classes). Indeed, for any representation a we can define the (N_c-dimensional) vectors:

$$[|a\rangle]_\mu = \sqrt{\frac{n_\mu}{N}}\chi_a(C_\mu) \text{ for } \mu = 1, ..., N_c\,. \tag{9.79}$$

There are N_r of these vectors for the N_r different irreps. It follows from Eq. (9.78) that this set of N_r vectors are all orthogonal. Since the maximum numbers of orthogonal vectors is N_c, we have

$$N_r \leq N_c\,. \tag{9.80}$$

That is, the number of representation is smaller or equal to the number of conjugation classes. This is the first step towards proving Lemma 9.7.2 (i.e. that the number of irreps is equal to the number of conjugacy classes) which we stated without proof earlier. In turns out this bound is tight (this is another consequence of the Grand Orthogonality Theorem - for a proof see Vincenzo Savona's notes on page 37) leading to Lemma 9.7.2.

9.9 Characters

We saw above that the traces of a representation of a group are useful. The set of traces associated with a representation are known as the *character* of the representation. Characters provide an elegant and systematic approach to analyzing and categorizing irreducible representations, as well as ascertaining the reducibility of a specific representation.

Definition 9.9.1 (Character)**.** The set of all traces $\{\chi_R(g)\}$ is called the character of the representation R.

As we saw above, two equivalent representations have the same character. Indeed if $R_2(g) = SR_1(g)S^{-1}$, then using the cyclic property of the trace we have $\mathrm{Tr}[R_2(g)] = \mathrm{Tr}[SR_1(g)S^{-1}] = \mathrm{Tr}[R_1(g)]$. In fact this is a sufficient condition as well:

Theorem 9.9.2 (Characters of Irreps)**.** *Two irreps are equivalent if and only if they have the same character.*

Demo. We already proved that the condition is necessary. To prove it is sufficient we reason by contradiction. Assume two irreps R_1 and R_2 are not equivalent but have the same character. Then using the petit Orthogonality theorem, we find that the sum of (modulus of) trace squared should be zero, which is impossible as the norm squared is positive and non-zero (the identity conjugacy class has trace 1). □

Or, turning it around, different (non-equivalent) irreps have different characters.

Using this approach, we can now compute degeneracy numbers for representations, that is compute how many copies of an irrep a given reducible representation contains. We first write:

$$R(g) = R_{1,1}(g) \oplus R_{1,2}(g) ... \oplus R_{1,b_1}(g) \oplus R_{2,1}(g) \oplus R_{2,2}(g) ... \oplus R_{2,b_2}(g) ... = \oplus_{a,x} R_{a,x}(g) \tag{9.81}$$

where $x = 1, ..., b_a$ with b_a denoting the degeneracy number. The question is how to find b_a? Using the characters of each irreps, we know that:

$$\chi_R(g) = \mathrm{Tr}\left[\begin{pmatrix} R_{1,1}(g) & 0 & 0 & 0 & \cdots \\ 0 & ... & 0 & 0 & \cdots \\ 0 & 0 & R_{1,b_1}(g) & 0 & \cdots \\ 0 & 0 & 0 & R_{2,1}(g) & \cdots \\ \cdots & & & & \end{pmatrix}\right] = \sum_a b_a \mathrm{Tr}[R_a(g)] = \sum_a b_a \chi_a(g)\,. \tag{9.82}$$

As the trace of all representations within the same conjugacy class are the same we can equivalently write

$$\chi_R(C_\mu) = \sum_a b_a \chi_a(C_\mu)\,. \tag{9.83}$$

We can combine this expression with the petite orthogonal theorem to find an expression for b_a. To do so we multiply by $n_\mu \chi_b^*(C_\mu)$, where n_μ is the number of element in class C_μ, and sum over classes

$$\sum_{\mu=1}^{N_c} n_\mu \chi_b^*(C_\mu)\chi_R(C_\mu) = \sum_{\mu=1}^{N_c} n_\mu \sum_a b_a \chi_b^*(C_\mu)\chi_a(C_\mu) \tag{9.84}$$

$$= \sum_a b_a \sum_{\mu=1}^{N_c} n_\mu \chi_b^*(C_\mu)\chi_a(C_\mu) = \sum_a b_a N \delta_{a,b} = N b_b \tag{9.85}$$

so that

$$b_a = \frac{1}{N}\sum_{\mu=1}^{N_c} n_\mu \chi_a^*(C_\mu)\chi_R(C_\mu)\,. \tag{9.86}$$

We thus now have a formula for each number of irreps contained in a given representation:

Theorem 9.9.3 (Computing Degeneracy)**.** *Assume a decomposition in irreps as*

$$R(g) = \oplus_{a,x} R_{a,x}(g) \tag{9.87}$$

for $x = 1, ..., b_a$. *Then we have*

$$b_a = \frac{1}{N}\sum_{\mu} n_\mu \chi_a^*(C_\mu)\chi_R(C_\mu) \tag{9.88}$$

Remember this formula! It will be very useful in the problem sheets this week.

Another interesting consequence of the petite orthogonal theorem is the following one:

Theorem 9.9.4 (Sufficient condition for irreps)**.** *A necessary and sufficient condition for a representation R to be an irrep is that*

$$\sum_{\mu=1}^{N_c} n_\mu |\chi(C_\mu)|^2 = N \tag{9.89}$$

Demo. Using Eq.(9.83) and the petit orthogonality theorem (Eq.(9.78)), we find that

$$\sum_{\mu=1}^{N_c} n_\mu |\chi(C_\mu)|^2 = \sum_{i,j} b_i b_j \sum_{\mu=1}^{N_c} n_\mu \chi_i(C_\mu)^* \chi_j(C_\mu) = N\sum_{i,j} b_i b_j \delta_{ij} = N\sum_i b_i^2 \tag{9.90}$$

Being irreducible means having only one of the b_i=1, which proves the theorem. □

For a finite group, it is easy to find the characters listed in table in the literature (google is your friend!), listed as follows:

irrep\ class	$C_1(e)$	C_2	C_3	C_4	C_5
R_1	1	1	1	1	1
R_2	d_2	$\chi_2(C_2)$	$\chi_2(C_3)$	$\chi_2(C_4)$	$\chi_2(C_5)$
R_3	d_3	$\chi_3(C_2)$	$\chi_3(C_3)$	$\chi_3(C_4)$	$\chi_3(C_5)$
R_4	d_4	$\chi_4(C_2)$	$\chi_4(C_3)$	$\chi_4(C_4)$	$\chi_4(C_5)$
R_5	d_5	$\chi_5(C_2)$	$\chi_5(C_3)$	$\chi_5(C_4)$	$\chi_5(C_5)$

Again, that was quite lot of quite technical material. And we've got more to come. So here's a panda (Fig 9.8). And if fluffy animals aren't your thing here's a clip of two guys trying to kayak down a melting ski slope.

9.9.1 Example with C3v.

Ok we now finally have the tools to put everything together and show how orthogonality relations can be used to identify irreps.

Let us again consider the C3v group, i.e. symmetry of the triangle. We first recall that it is a non-Abelian group of order 6. The conjugacy classes are $C_e = \{e\}, C_1 = \{c_+, c_-\}$ and $C_2 = \{\sigma, \sigma', \sigma''\}$ and so, as we saw before, from Lemma (9.7.2) there can be only 3 irreps.

We saw the 2D irrep in Eq. (9.61):

$$\begin{aligned} R(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ R(c_+) &= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, R(c_-) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ R(\sigma) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, R(\sigma') = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, R(\sigma'') = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \end{aligned} \tag{9.91}$$

There we simply claimed that this was an irrep. Now we can use Theorem 9.9.4 to check. Namely we have,

$$\sum_{\mu=1}^{N_c} n_\mu |\chi(C_\mu)|^2 = 1 \times 2^2 + 2 \times (-1)^2 + 3 \times 0 = 6 = N\,. \tag{9.92}$$

What are the other irreps? We can of course have the trivial irrep where every group element is represented by a scalar equal to one. The trivial 1D irrep:

$$R(e) = 1, R(c_+) = 1, R(c_-) = 1, R(\sigma) = 1, R(\sigma') = 1, R(\sigma'') = 1 \tag{9.93}$$

(This is indeed an irreducible representation as $1 + 2 \times 1 + 3 \times 1 = 6$ in line with Theorem 9.9.4).

Now to identify the missing irrep. From Burnside's Lemma we know that it has to be 1D (i.e, $1^2 + 2^2 + l^2 = 6$ implies $l = 1$). From the petit orthogonality theorem we know that the characters of this final representation must be orthogonal. Thus denote the characters of the

Figure 9.8: **Motivational Panda.** Even if you're struggling a little to follow by this point you're still doing better than this panda. (God knows how these animals survive in the wild).

missing representation as $(\chi_e, \chi_c, \chi_c, \chi_\sigma, \chi_\sigma, \chi_\sigma)$ we have $(1,1,1,1,1,1).(\chi_e, \chi_c, \chi_c, \chi_\sigma, \chi_\sigma, \chi_\sigma) = \chi_e + 2\chi_c + 3\chi_\sigma = 0$ and $(2,-1,-1,0,0,0).(\chi_e, \chi_c, \chi_c, \chi_\sigma, \chi_\sigma, \chi_\sigma) = 2\chi_e - 2\chi_c = 0$. Thus we have $\chi_e = \chi_c$ and $\chi_\sigma = -\chi_e$. The only 1D representation that satisfies these conditions and Lemma (9.7.2) is thus:

$$R(e) = 1, R(c_+) = 1, R(c_-) = 1, R(\sigma) = -1, R(\sigma') = -1, R(\sigma'') = -1 \tag{9.94}$$

(Check for yourself that this is indeed an irrep for C3v!)

Thus for the character table for the group C3v we have Table 9.17.

	e	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

Table 9.1: Character table for point group C3v. Here $A1$ and $A2$ deontes the 1D representation in Eq. (9.93) and Eq. (9.94), and E denotes the $2D$ representation in Eq.(9.61).

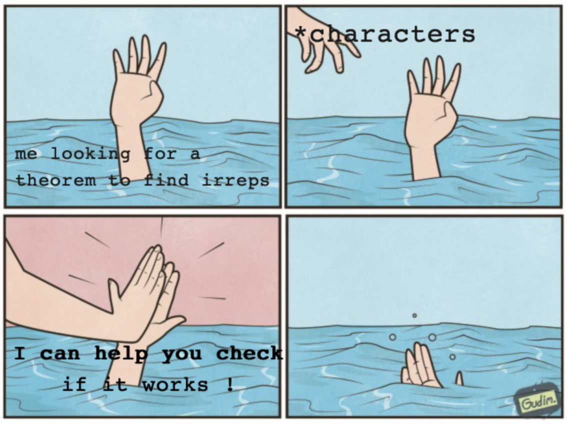

Figure 9.9: Note that in the above example we could get away with just studying the characters and the petite orthogonality theorem to identify our irreps. However, in general the characters will not suffice and you'll have to have already identified some non-trivial irreps and then can use the grand orthogonality theorem to help you identify the remainders. That said, even in this case knowing the character at least helps you guess the diagonal of your irrep. Credit: Mehdi Haddad.